Phase space analysis of Hermite-type semigroups

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Pseudo-Differential Operators and Related Topics

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The free fractional Schrödinger flow

Given $\nu > 0$, consider the Cauchy problem

$$\begin{cases} i\partial_t u(t,x) + (-\Delta_x)^\nu u(t,x) = 0 \\ u(0,x) = u_0(x), \end{cases} \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d,$$

where $\Delta_x = \sum_{j=1}^{d} \partial_{x_j}^2$ is the standard Laplacian operator on $\mathbb{R}^d$.

A well-known result

The free fractional Schrödinger flow $e^{-it(-\Delta)^\nu}$ is a Fourier multiplier with symbol $m(\xi) = e^{-it|\xi|^{2\nu}}$.

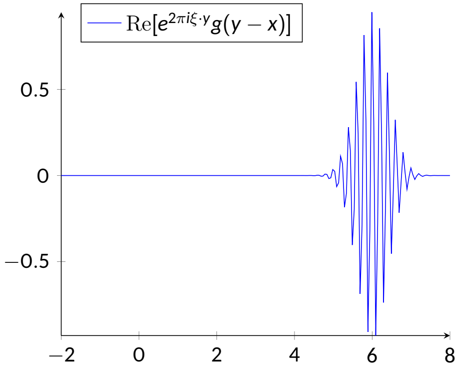
In general, it fails to be bounded on $L^p(\mathbb{R}^d)$ except for $p = 2$.

Are there "better" spaces where to frame this problem?

Phase space analysis via Gabor wave packets

Decomposition of a "signal" $f \in \mathcal{S}'(\mathbb{R}^d)$ along **Gabor wave packets**

$$\pi(x,\xi)g(y) = (2\pi)^{-\frac{d}{2}} e^{i\xi\cdot y} g(y-x), \quad g \in \mathcal{S}(\mathbb{R}^d),$$



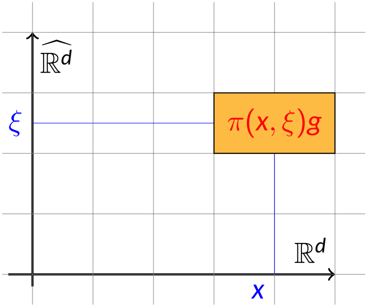
Phase space analysis via Gabor wave packets

Decomposition of a "signal" $f \in \mathcal{S}'(\mathbb{R}^d)$ along Gabor wave packets

$$\pi(x,\xi)g(y) = (2\pi)^{-\frac{d}{2}} e^{i\xi\cdot y} g(y-x), \quad g \in \mathcal{S}(\mathbb{R}^d),$$

is known as the **Gabor transform** (or short-time Fourier transform):

$$V_g f(x,\xi) := \langle f, \pi(x,\xi)g\rangle = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\xi\cdot y} f(y)\,\overline{g(y-x)}\,dy, \quad (x,\xi) \in \mathbb{R}^{2d}.$$



Modulation spaces

For $1 \leq p, q \leq \infty$ we introduce the **modulation spaces** [Feichtinger 1981]

$$M^{p,q}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{M^{p,q}} < \infty \right\},$$

where

$$\|f\|_{M^{p,q}} := \left\| \|V_g f(x,\xi)\|_{L^p_x} \right\|_{L^q_\xi}.$$

Increasing Banach spaces: if $p_1 \leq q_2$ and $p_2 \leq q_2$ then $M^{p_1,q_1} \subseteq M^{p_2,q_2}$.

For $s \in \mathbb{R}$ we also introduce the **weighted** space $M^{p,q}_s(\mathbb{R}^d)$, with

$$\|f\|_{M^{p,q}_s} := \left\| \|V_g f(x,\xi)\langle(x,\xi)\rangle^s\|_{L^p_x} \right\|_{L^q_\xi},$$

where $\langle y\rangle^s := (1+|y|^2)^{s/2}$, $y \in \mathbb{R}^n$. **Some remarks:**

- $\bigcap_{s\geq 0} M^{p,q}_s(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d), \qquad \bigcup_{s\geq 0} M^{p,q}_{-s}(\mathbb{R}^d) = \mathcal{S}'(\mathbb{R}^d).$
- $\left\| \|V_g f(x,\xi)\langle\xi\rangle^s\|_{L^2_x} \right\|_{L^2_\xi} \asymp \|\hat{f}(\xi)\langle\xi\rangle^s\|_{L^2_\xi} = \|f\|_{H^s}.$

Fourier multipliers on modulation spaces

[Bényi et al. — *JFA* 2007] [Chen et al. — *Nonlinear Anal.* 2012]

Given $t > 0$, the Fourier multipliers with symbols

$$a_t(\xi) = e^{-it|\xi|^\alpha} \quad (0 \le \alpha \le 2), \qquad b_t(\xi) = e^{-t|\xi|^\beta} \quad (\beta > 0),$$

$$c_t(\xi) = \frac{\sin(t|\xi|^\gamma)}{|\xi|^\delta} \quad (0 < \delta \le \gamma \le 1),$$

are bounded on any modulation space $M^{p,q}(\mathbb{R}^d)$, $1 \le p, q \le \infty$.

$\rightsquigarrow$ fractional Schrödinger/heat, wave equation, etc.

Beyond the free particle

The **Hermite operator** in $\mathbb{R}^d$:

$$\mathcal{H} := -\Delta + |x|^2 = \sum_{j=1}^{d} \left(-\partial_{x_j}^2 + x_j^2 \right).$$

Also known as the **quantum harmonic oscillator**,

that is the PSDO with quadratic symbol

$$H(x,\xi) = |x|^2 + |\xi|^2, \qquad (x,\xi) \in \mathbb{R}^{2d},$$

the paramount example of a **globally (hypo-)elliptic** symbol:

$$H(x,\xi) \gtrsim 1 + |x|^2 + |\xi|^2 \qquad \text{if} \qquad |x|^2 + |\xi|^2 \gtrsim 1.$$

Spectral structure for the Hermite operator

The spectral decomposition of $\mathcal{H}$ in $L^2(\mathbb{R}^d)$ is given by

$$\mathcal{H} = \sum_{k=0}^{\infty} (2k+d) P_k, \qquad P_k = \sum_{|\alpha|=k} \langle \cdot, \Phi_\alpha \rangle \Phi_\alpha \quad : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d),$$

where $\{\Phi_\alpha : \alpha \in \mathbb{N}^d\}$ are orthonormal Hermite functions:

$$\Phi_\alpha(x) = C_d(\alpha)\, e^{\frac{|x|^2}{2}}\, \partial_x^\alpha e^{-|x|^2}, \qquad x \in \mathbb{R}^d.$$

Spectral multipliers and associated semigroups

The **spectral multiplier** with symbol $m\colon \mathbb{N} \to \mathbb{C}$ is defined by

$$m(\mathcal{H})f := \sum_{k\geq 0} m(2k+d)P_k f, \qquad f \in \mathcal{S}'(\mathbb{R}^d).$$

- The Schrödinger semigroup: $e^{-it\mathcal{H}}$

  $\rightsquigarrow$ is a metaplectic operator, hence: $\|e^{-it\mathcal{H}}f\|_{M^{p,p}} \lesssim \|f\|_{M^{p,p}}$.

- **The heat semigroup: $e^{-t\mathcal{H}}$**

  $\rightsquigarrow$ Bounded on $L^p(\mathbb{R}^d)$, what about its phase space properties?

Fractional heat flows for the harmonic oscillator

Problem. $e^{-t\mathcal{H}^\nu} = \sum_{k\geq 0} e^{-t(2k+d)^\nu} P_k$ is **not** a Fourier multiplier!

Moreover, no explicit representation formulae for $e^{-t\mathcal{H}^\nu}$ in general.

Solution? $\mathcal{H} \quad = \quad \mathrm{op_w}(H) \quad \rightsquigarrow \mathcal{H}^\nu ? \quad \rightsquigarrow e^{-t\mathcal{H}^\nu} ?$

$\rightsquigarrow$ Spectral theory of **powers and heat kernels of globally elliptic PSDOs**

[Shubin 1978 — Helffer 1984 — Nicola, Rodino 2010]

Phase space analysis of Shubin symbols

For $s \in \mathbb{R}$ consider the **Shubin classes**

$$\Gamma^s(\mathbb{R}^{2d}) := \{a \in C^\infty(\mathbb{R}^{2d}) : |\partial^\alpha a(z)| \lesssim_\alpha \langle z \rangle^{s-|\alpha|}, \quad z \in \mathbb{R}^{2d}, \alpha \in \mathbb{N}^{2d}\}.$$

It is clear that $H(z) = |z|^2 \in \Gamma^2$.

Phase space regularity of Shubin operators

If $a \in \Gamma^s$ then $\mathrm{op_w}(a) \colon M^{p,q}_s \to M^{p,q}$ is bounded for all $1 \le p, q \le \infty$.

Intuition. Read the phase space effect of $\mathrm{op_w}(a)$: for $g, \gamma \in \mathcal{S}(\mathbb{R}^d)$,

$$V_\gamma(\mathrm{op_w}(a)f)(w) = \int_{\mathbb{R}^{2d}} K(w,z) V_g f(z) dz, \qquad w, z \in \mathbb{R}^{2d},$$

where the **Gabor kernel** satisfies, for every $N \in \mathbb{N}$,

$$|K(w,z)| := |\langle \mathrm{op_w}(a)\pi(z)g, \pi(w)\gamma \rangle| \lesssim \langle w+z \rangle^s \langle w-z \rangle^{-2N}.$$

Symbols of some Hermite-type operators

The symbol of the fractional Hermite operator $\mathcal{H}^\nu$

For $\nu \in \mathbb{R}$, $\mathcal{H}^\nu$ is a PSDO with Weyl symbol $H_\nu \in \Gamma^{2\nu}$.

If $\nu > 0$ there exists $r \in \Gamma^{2\nu-2}$ such that

$$H_\nu(x,\xi) = (|x|^2 + |\xi|^2)^\nu + r(x,\xi), \qquad |x|^2 + |\xi|^2 \geq 1.$$

The symbol of the fractional heat operator $e^{-t\mathcal{H}^\nu}$

For $\nu > 0$ and $t \in [0,T]$, $e^{-t\mathcal{H}^\nu}$ is a PSDO with Weyl symbol $h_\nu(t)$ s.t.

$t^N h_\nu(t)$ belongs to a bounded set of $\Gamma^{-2\nu N}$ for any $N \in \mathbb{N}$.

Fractional heat flows on modulation spaces

[Bhimani, Manna, Nicola, Thangavelu, T. — *Adv. Math.* 2021]

Let $\nu, t > 0$ and assume $p_2 \le p_1, q_2 \le q_1$.

Then $e^{-t\mathcal{H}^\nu} : M^{p_1,q_1}(\mathbb{R}^d) \to M^{p_2,q_2}(\mathbb{R}^d)$ is bounded, with

$$\|e^{-t\mathcal{H}^\nu}\|_{M^{p_1,q_1}\to M^{p_2,q_2}} \lesssim C(t) = \begin{cases} e^{-td^\nu} & (t \ge 1) \\ t^{-\sigma} & (0 < t \le 1), \end{cases}$$

$$\text{where} \quad \sigma := \frac{d}{2\nu}\Big(\frac{1}{p_2} - \frac{1}{p_1} + \frac{1}{q_2} - \frac{1}{q_1}\Big) \ge 0.$$

Sharp for $t \ge 1$: $e^{-t\mathcal{H}^\nu}\Phi_0 = e^{-td^\nu}\Phi_0$ (Gaussian ground state).

Intuition. Phase-space effect of $e^{-t\mathcal{H}^\nu}$ is roughly multiplication by $F_t(x,\xi) = e^{-t(|x|^2+|\xi|^2)^\nu}$, and actually $\|\cdot F_t\|_{L^{p_1,q_1}\to L^{p_2,q_2}} \asymp C(t)$.

Given $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}_+$, consider the Cauchy problem

$$\partial_t u(t,x) + \mathcal{H}^\nu u(t,x) = \lambda |u|^{2k} u, \qquad u(0,x) = u_0(x),$$

with $(t,x) \in [0,+\infty) \times \mathbb{R}^d$. If $q' \ge \max(2k+1, kd/\nu)$:

- **(Local WP)** If $t \in [0,T]$ then $\|u(t,\cdot)\|_{M^{p,q}} \le C(T)\|u_0\|_{M^{p,q}}$.
- **(Global WP)** If $\|u_0\|_{M^{p,q}} < \varepsilon$ is small enough then there is a unique *global* solution $u \in L^\infty([0,+\infty); M^{p,q})$ s.t. $\|u(t,\cdot)\|_{M^{p,q}} \lesssim e^{-td^\nu}$.

- Example of allowed initial datum (for suitably small parameters):

$$u_0(x) = \alpha |x|^{-\beta}(1 + \gamma \cos |x|^2) \quad (\notin L^p).$$

- **No blow up in finite time** as in the standard (focusing) heat equation for arbitrarily small constant initial data (hence in $M^{\infty,1}$).

Fractional heat flows on Lebesgue spaces

[Bhimani, Manna, Nicola, Thangavelu, T. — *Fract. Calc. Appl. Anal.* 2023]

Let $t, \nu > 0$ and $1 < p, q < \infty$, or $(p,q) = (1,\infty)$ or $(p,q) = (\infty, 1)$.

Set $\sigma := \frac{d}{2\nu}\left|\frac{1}{p} - \frac{1}{q}\right|$. Then $e^{-t\mathcal{H}^\nu} : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is bounded, with

$$\|e^{-t\mathcal{H}^\nu}\|_{L^p \to L^q} \lesssim C(t) = \begin{cases} e^{-td^\nu} & (t \ge 1) \\ t^{-\sigma} & (0 < t \le 1). \end{cases}$$

The result holds for all $1 \le p, q \le \infty$ if $0 < \nu \le 1$.

Tools. Embeddings $L^p \hookrightarrow M^{p,\infty}$ and $M^{q,1} \hookrightarrow L^q$ $(t > 1)$ + refined analysis of the symbol $(0 < t < 1)$ + subordination $(0 < \nu \le 1)$

Applications. Local and global nonlinear wellposedness (critical exponent)
$\rightsquigarrow$ Strichartz estimates (TT^*)

The twisted Laplacian (special Hermite operator)

Setting $z = (x, y) = \mathbb{R}^{2d}$ consider [Strichartz 1989]

$$\mathcal{L} = -\sum_{j=1}^{d} \left[\left(\partial_{x_j} - \frac{i}{2} y_j \right)^2 + \left(\partial_{y_j} + \frac{i}{2} x_j \right)^2 \right]$$
$$= -\Delta_z + \frac{1}{4} |z|^2 - i \sum_{j=1}^{d} \left(x_j \partial_{y_j} - \partial_{x_j} y_j \right).$$

Landau Hamiltonian: q. harmonic oscillator + uniform magnetic field.

The Weyl symbol fails to be globally (hypo-)elliptic, but

$\mathcal{L}$ is **globally regular**: if $f \in \mathcal{S}'(\mathbb{R}^{2d})$ and $\mathcal{L}f \in \mathcal{S}(\mathbb{R}^{2d})$ then $f \in \mathcal{S}(\mathbb{R}^{2d})$.

Now what?

Spectral structure of the special Hermite operator

$\rightsquigarrow$ Spectral multipliers for $\mathcal{L}$: [Thangavelu 1991 — Koch, Ricci 2007]

$$m(\mathcal{L}) = \sum_{k\geq 0} m(2k+d)Q_k, \quad Q_k f = \sum_{\substack{\alpha\in\mathbb{N}^d \\ |\beta|=k}} \langle f, \Phi_{\alpha,\beta}\rangle \Phi_{\alpha,\beta} = \boldsymbol{f} \times \boldsymbol{\varphi_k},$$

where $\Phi_{\alpha,\beta} \propto V_{\Phi_\beta}\Phi_\alpha$ (**special Hermite functions**)

and $\times\varphi_k$ is the **twisted convolution** with a Laguerre function:

$$(a \times b)(z) := \pi^{-d} \int_{\mathbb{R}^{2d}} e^{2i\sigma(z,w)} a(z-w)b(w)dw, \quad \sigma(z,w) = Jz \cdot w.$$

Connection with product of symbols via symplectic Fourier transform:

$$\mathcal{F}_\sigma(a\#b) = \mathcal{F}_\sigma a \times \mathcal{F}_\sigma b, \quad \text{where} \quad \mathcal{F}_\sigma(f)(\zeta) = \pi^{-d} \int_{\mathbb{R}^{2d}} e^{-2i\sigma(\zeta,z)} f(z)dz.$$

A bridge towards the harmonic oscillator

$\mathcal{L}$ as the **Landau-Weyl phase space quantization** of $H(q,p) = q^2 + p^2$:
[de Gosson 2008 — Buzano, Oliaro 2020]

$$q_j \mapsto -i\partial_{x_j} - \frac{1}{2}y_j, \qquad p_j \mapsto -i\partial_{y_j} + \frac{1}{2}x_j.$$

CCR $\rightsquigarrow$ $\mathcal{L}$ is unitarily equivalent to a partial harmonic oscillator:

$$\boxed{\mathcal{A}_J\mathcal{L} = (I \otimes \mathcal{H})\mathcal{A}_J,}$$

where $$\mathcal{A}_J f(x,y) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot u} f\left(u+\frac{y}{2}, u-\frac{y}{2}\right) du, \quad (x,y) \in \mathbb{R}^{2d}.$$

In fact, $\mathcal{A}_J$ is a metaplectic operator [Gramchev, Pilipović, Rodino 2009–10]

Towards transference principles

We write M^p_s for $M^{p,p}_s$.

Transference via Schur-type interpolation [T. — 😊 2306.00592]

If $m(\mathcal{H})$ is bounded $M^1_s(\mathbb{R}^d) \to M^1_r(\mathbb{R}^d)$ and $M^\infty_s(\mathbb{R}^d) \to M^\infty_r(\mathbb{R}^d)$,

then $m(\mathcal{L})$ is bounded $M^p_s(\mathbb{R}^{2d}) \to M^p(\mathbb{R}^{2d})$ for **all** $1 \le p \le \infty$, with

$$\|m(\mathcal{L})\|_{M^p_s \to M^p} \lesssim \|m(\mathcal{H})\|^{1/p}_{M^1_s \to M^1_r} \|m(\mathcal{H})\|^{1/p'}_{M^\infty_s \to M^\infty_r}.$$

Boundedness cannot be transferred on $M^{p,q}$ if $p \neq q$.

$\rightsquigarrow \mathcal{L}^\nu : M^p_{\max\{2\nu,0\}}(\mathbb{R}^{2d}) \to M^p(\mathbb{R}^{2d})$ with $\nu \in \mathbb{R}$.

$\rightsquigarrow \|\mathcal{L}^{-\delta/2} e^{it\mathcal{L}^{\gamma/2}}\|_{M^p \to M^p} \le C(T)$ if $t \in [0,T]$ and $\gamma \le 1$, $\delta \ge 0$

Beyond transference by exploiting twistedness

Fractional heat flow for $\mathcal{L}$ [T. — 2306.00592]

Let $t > 0$, $0 < \nu \leq 1$ and $1 \leq p_1, p_2, q_1, q_2 \leq \infty$. Then

$e^{-t\mathcal{L}^\nu} : M^{p_1,q_1}(\mathbb{R}^{2d}) \to M^{p_2,q_2}(\mathbb{R}^{2d})$ is bounded **if and only if $q_2 \geq q_1$**:

$$\|e^{-t\mathcal{L}^\nu}\|_{M^{p_1,q_1}\to M^{p_2,q_2}} \lesssim C(t) = \begin{cases} e^{-td^\nu} & (t \geq 1) \\ t^{-\lambda} & (0 < t \leq 1), \end{cases}$$

$$\text{where} \quad \lambda := \max\left\{ \frac{d}{\nu}\left(\frac{1}{p_2} - \frac{1}{p_1}\right), 0 \right\}.$$

Tools. $e^{-t\mathcal{L}} f = f \times (16\pi \sinh t)^{-d} e^{-\coth(t)|\cdot|^2/4}$ [Wong 2005]

+ analysis of twisted convolution and Weyl product on modulation spaces [Cordero, Holst, Toft, Wahlberg 2010–14]

+ Bochner subordination ($0 < \nu < 1$).

Thanks for your attention!