Existence of concentration optimizers in phase space

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Gabor wave packets

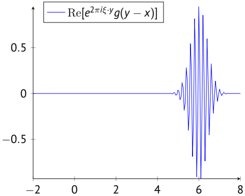
A **Gabor atom** is a function of the type

$$\pi(x,\xi)g(y) = e^{2\pi i\xi\cdot y}g(y-x),$$

where

- $g \in L^2(\mathbb{R}^d)$ is a function possessing **good localization in phase space** $\mathbb{R}^d \times \widehat{\mathbb{R}^d} \simeq \mathbb{R}^{2d}$ – e.g., a Gaussian function $g(y) = e^{-\pi|y|^2}$
- $\pi(x,\xi) = M_\xi T_x$ is the **phase space shift** along $(x,\xi) \in \mathbb{R}^{2d}$, with

$$M_\xi f(y) = e^{2\pi i\xi\cdot y}f(y), \quad T_x f(y) = f(y-x), \quad y \in \mathbb{R}^d.$$

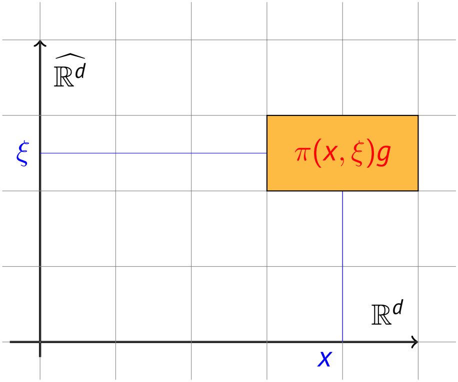


$$(x = 6, \quad \xi = 5)$$

Gabor analysis of functions

Decomposition of $f \in L^2(\mathbb{R}^d)$ along Gabor wave packets is known as the **Gabor transform** (or **short-time Fourier transform**):

$$V_g f(x,\xi) := \langle f, \pi(x,\xi) g \rangle = \int_{\mathbb{R}^d} e^{-2\pi i y \cdot \xi} f(y) \overline{g(y-x)}\, dy, \quad (x,\xi) \in \mathbb{R}^{2d}.$$



The choice of the atom

Fixed Gaussian window $\rightsquigarrow$ FBI tr., Bargmann-Fock-Segal tr., . . .

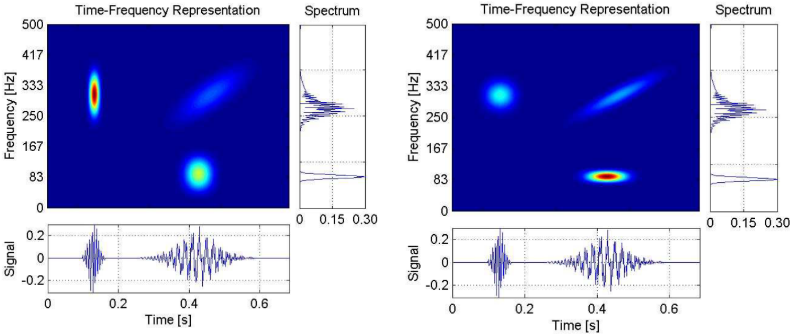


Figure: Plot of $|V_g f|^2$ with narrow vs wide time Gaussian window g
DOI: 10.2478/mms-2014-0054

TFA without auxiliary functions?

"Pure" phase space representation of $f \rightsquigarrow$ **quadratic transforms**

The standard pathway: from $L(f,g)$ to $Q(f) := L(f,f)$.

Problem: **cross interferences** do appear!

$$Q(\alpha_1 f_1 + \alpha_2 f_2) = |\alpha_1|^2 Q(f_1) + \alpha_1 \overline{\alpha_2} L(f_1, f_2) + \overline{\alpha_1} \alpha_2 L(f_2, f_1) + |\alpha_2|^2 Q(f_2).$$

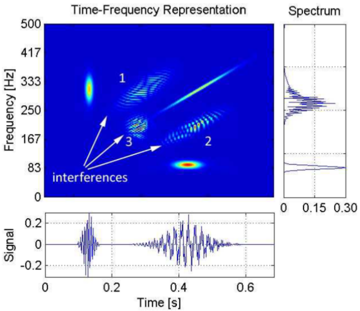


Figure: DOI: 10.2478/mms-2014-0054

The ambiguity distribution

Consider the **ambiguity transform** of $f \in L^2(\mathbb{R}^d)$:

$$Af(x,\xi) := e^{\pi i x\cdot\xi} V_f f(x,\xi) = \int_{\mathbb{R}^d} e^{-2\pi i y\cdot\xi} f(y+x/2)\overline{f(y-x/2)}dy.$$

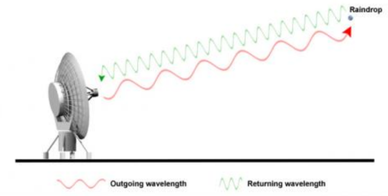
In general, the **cross-ambiguity transform** of $f, g \in L^2(\mathbb{R}^d)$ is

$$A(f,g)(x,\xi) := \int_{\mathbb{R}^d} e^{-2\pi i y\cdot\xi} f(y+x/2)\overline{g(y-x/2)}dy.$$

Nice properties of the ambiguity function:

- $Af \in C(\mathbb{R}^{2d})$ and $\lim\limits_{|(x,\xi)|\to\infty} |Af(x,\xi)| = 0$
- $\|Af\|_\infty = \max\limits_{(x,\xi)\in\mathbb{R}^{2d}} |Af(x,\xi)| = Af(0,0) = \|f\|_{L^2}^2 = \|Af\|_{L^2}$

Example (RADAR vs UFO [Gröchenig – *Found. TFA*, 2001])



Send a pulse

$$f(t) = e^{2\pi i \omega_0 \cdot t}\sigma(t),$$

$\mathrm{supp}(\hat{f}) \subseteq [\omega_0 - A, \omega_0 + A]$, $A/\omega_0 \ll 1$.

Echo travels (twice) a distance ℓ at speed $c \Rightarrow$ **time lag** $\Delta t \approx 2\ell/c$.

Doppler effect $\Rightarrow$ **frequency shift** $\Delta\omega \approx -2\omega_0 v/c$.

Received echo has the form $e = M_{\Delta\omega} T_{\Delta t} f$

$\Rightarrow$ compare with TF shifts of f: $|\langle e, M_\omega T_t f\rangle| = |Af(t - \Delta t, \omega - \Delta\omega)|$.

$\max_{(x,\xi)\in\mathbb{R}^{2d}} |Af(x,\xi)| = Af(0,0) \Rightarrow$ empirical estimate of $\Delta t, \Delta\omega$.

Desirable ambiguity profiles

The RADAR example shows that one would like to design pulses f with "thumbtack" ambiguity distribution near the origin.

Problem: the uncertainty principle

Arbitrarily high concentration in phase space is forbidden.

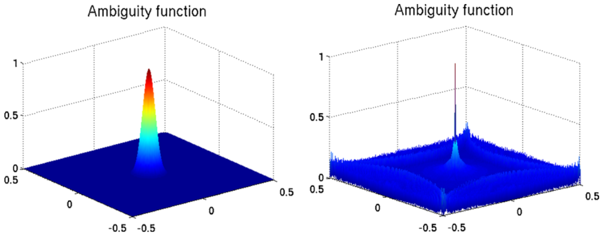


Figure: Af for Gaussian (left) and high-order Hermite functions (right)
DOI: 10.1007/s10444-013-9323-2

What about the ambiguity concentration?

Given a phase space subset $\Omega \subset \mathbb{R}^{2d}$ of finite (non-zero) measure and $1 \le p \le \infty$, consider the L^p ambiguity concentration measure

$$\|Af\|_{L^p(\Omega)} = \left(\int_\Omega |Af(x,\xi)|^p dx d\xi \right)^{1/p}.$$

Problem. Are there functions that attain the optimal concentration

$$\sup \left\{ \|Af\|_{L^p(\Omega)} : \|f\|_{L^2} = 1 \right\}?$$

Direct method of calc. var. for existence of optimizers of $F\colon X \to \mathbb{R}$:

(sequential) compactness of X

+

(semi)continuity of F

Enemy #1 – defect of compactness (and its source)

Consider the time-frequency shifts $\{\pi(z) = M_\omega T_t : z = (t, \omega) \in \mathbb{R}^{2d}\}$.

For any $f \in L^2(\mathbb{R}^d)$ and any sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^{2d}$ such that $|z_n| \to +\infty$, we have

$$\pi(z_n)f \to 0 \quad \text{weakly in } L^2 \quad \text{as } n \to +\infty$$

– indeed, recall that $|\langle \pi(z_n)f, g\rangle| = |V_g f(-z_n)|$ and $V_g f \in C_0(\mathbb{R}^{2d})$.

Problem: the functional is **invariant** under TF shifts!

$$|A(\pi(z)f)(x,\xi)| = |Af(x,\xi)|, \qquad \forall z \in \mathbb{R}^{2d},\ (x,\xi) \in \mathbb{R}^{2d}.$$

Loss of compactness by non-compact group actions

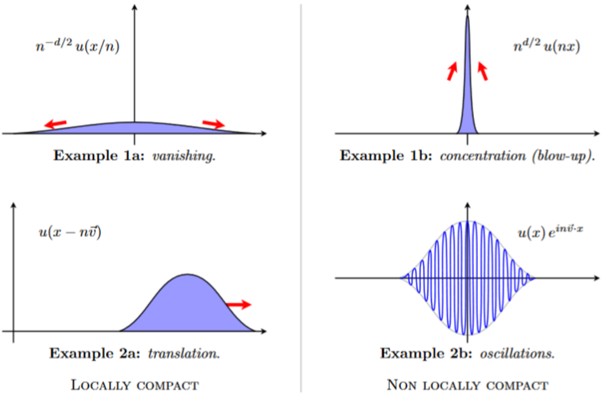


FIGURE 1. Four typical behaviors of a sequence $\{u_n\} \subset L^2(\mathbb{R}^d)$ with $u_n \rightharpoonup 0$ while $\int_{\mathbb{R}^d} |u_n|^2 = \lambda$ for all n. In the examples, u is a fixed (smooth) function in $L^2(\mathbb{R}^d)$, such that $\int_{\mathbb{R}^d} |u|^2 = \lambda$.

Figure: M. Lewin via hal-02450559

Enemy #2 – failure of semicontinuity

Fact 1. If g is a Gaussian function and $|\Omega| > 0$, then $\|Ag\|_{L^2(\Omega)} > 0$.

Fact 2. An **asymptotic decoupling** estimate of Brezis-Lieb type holds:

for every $f \in L^2(\mathbb{R}^d)$ and sequence $(z_n)_{n\in\mathbb{N}} \subset \mathbb{R}^{2d}$ s. t. $|z_n| \to +\infty$,

$$\lim_{n\to+\infty} \|A(f + \pi(z_n)g)\|^2_{L^2(\Omega)} = \|Af\|^2_{L^2(\Omega)} + \|Ag\|^2_{L^2(\Omega)}.$$

As a result, we obtain the lack of sequential weak upper semicontinuity of the functional $f \mapsto \|Af\|_{L^2(\Omega)}$:

$$\pi(z_n)g \rightharpoonup 0 \quad \text{but} \quad \lim_{n\to+\infty} \|A(f + \pi(z_n)g)\|^2_{L^2(\Omega)} > \|Af\|^2_{L^2(\Omega)}.$$

Nevertheless... good news!

Existence of optimizers **[Nicola, Romero, T. – *Calc. Var.* 2023]**

Let $\Omega \subset \mathbb{R}^{2d}$ be a measurable subset of finite measure $|\Omega| > 0$.

- **Existence of an ambiguity concentration optimizer**

  If $1 \le p < \infty$, there exists $\tilde{f} \in L^2(\mathbb{R}^d)$ with $\|\tilde{f}\|_{L^2} = 1$ such that

$$\|A\tilde{f}\|_{L^p(\Omega)} = \sup\{\|Af\|_{L^p(\Omega)} : \|f\|_{L^2} = 1\}.$$

- **Approaching the optimizer via (sub)sequences**

  If $1 < p < \infty$, for every (normalized) maximizing sequence $f^{(n)}$ there exist $z^{(n)} \in \mathbb{R}^{2d}$ such that, passing to subsequences,

$$\pi(-z^{(n)})f^{(n)} \to \tilde{f} \quad \text{in } L^2.$$

Dealing with defects of compactness

We resort to ideas and techniques of **concentration compactness**[1]: consider again the optimization of a functional $F\colon X \to \mathbb{R}$.

- **Tradeoff** between a "strong topology" (with few compact sets) and a "weak topology" (with few continuous functionals)
- CC phenomenon: there is an **"intermediate topology"** where F is continuous but X is not s. compact, yet any sequence $x^{(n)}$ converges "intermediately" to a sequence $y^{(n)} = \sum_j y_j^{(n)}$ that consists *only* of profiles $y_j^{(n)}$ causing lack of compactness.

[1] Sacks-Uhlenberg (1981), Brezis-Nirenberg (1983), Lions (1984), Struwe (1984), Brezis-Coron (1985), Lions (1985), Tintarev-Fieseler (2007, 2020).

A toy model – X: unit sphere in $\ell^2(\mathbb{Z})$

"Strong top." by ℓ^2 norm convergence: $x^{(n)} \to 0$ iff $\|x^{(n)}\|_{\ell^2} \to 0$.

"Weak top." by weak convergence: $x^{(n)} \rightharpoonup 0$ iff $\langle x^{(n)}, u\rangle \to 0\ \forall u \in \ell^2$.

X is invariant under shifts T^n, $n \in \mathbb{Z}$, and also not weakly closed.

"Intermediate top." by ℓ^∞ norm convergence – indeed, one has

$$\|x^{(n)}\|_{\ell^\infty} \to 0 \quad \Longleftrightarrow \quad \langle T^{h^{(n)}} x^{(n)}, u\rangle \to 0 \quad \forall u \in \ell^2, (h^{(n)})_n \subset \mathbb{Z}.$$

The intermediate topology is thus equivalent to the one induced by **D-weak convergence** associated with translation group (**dislocations**).

Travelling profiles in $\ell^2(\mathbb{Z})$

The travelling **profiles** $T^{h^{(n)}}x^{(n)}$ are the source of lack of weak comp.

Isolating dislocated profiles leads to a refinement of Banach-Alaoglu theorem, namely **profile decompositions** of a sequence $x^{(n)} \in X$:

$$x^{(n)} = \sum_{j=1}^{k} T^{h_j^{(n)}} x_j + w_k^{(n)}, \qquad k \geq 1,$$

- **Asymptotic orthogonality**: $|h_j^{(n)} - h_{j'}^{(n)}| \to \infty$ as $n \to \infty, \quad j \neq j'$,
- **Conservation of mass**: $\sum_{j=1}^{k} \|x_j\|_{\ell^2}^2 + \limsup_{n\to\infty} \|w_k^{(n)}\|_{\ell^2}^2 \leq 1$,
- **Vanishing IT remainder**: $\lim_{k\to\infty} \limsup_{n\to\infty} \|w_k^{(n)}\|_{\ell^\infty} = 0$.

Time-frequency profile decompositions in $L^2(\mathbb{R}^d)$

Dislocations: time-frequency shifts $\{\pi(z) : z \in \mathbb{R}^{2d}\}$

D-weak convergence: uniform convergence of Gabor transforms

$$f^{(n)} \to_D f \iff \sup_{z\in\mathbb{R}^{2d}} |\langle f^{(n)} - f, \pi(z)g\rangle| = \|V_g(f^{(n)} - f)\|_\infty \to 0 \quad \forall g \in L^2.$$

Profile decompositions: if $\limsup_{n\to\infty} \|f^{(n)}\|_{L^2} \le 1$,

$$f^{(n)} = \sum_{j=1}^{k} \pi(z_j^{(n)}) f_j + w_k^{(n)}, \qquad k \ge 1.$$

Step 1 – ambiguity and profiles

Step 0: $L := \sup \left\{ \frac{\|Af\|_{L^p(\Omega)}}{\|f\|_{L^2}^2} : f \in L^2(\mathbb{R}^d) \setminus \{0\} \right\} \Longrightarrow 0 < L \le |\Omega|^{1/p}$.

Let $f^{(n)}$ be a L^2-normalized maximizing sequence – $\|f^{(n)}\|_{L^2} = 1$.
After passing to a subsequence, consider profile decompositions:

$$f^{(n)} = F_k^{(n)} + w_k^{(n)}, \qquad F_k^{(n)} := \sum_{j=1}^{k} \pi\big(z_j^{(n)}\big) f_j$$

The sesquilinearity of the ambiguity function yields

$$A(f^{(n)}) = \sum_{j=1}^{k} A(\pi(z_j^{(n)}) f_j) + \sum_{\substack{1 \le j, j' \le k \\ j \ne j'}} A(\pi(z_j^{(n)}) f_j, \pi(z_{j'}^{(n)}) f_{j'})$$
$$+ A(F_k^{(n)}, w_k^{(n)}) + A(w_k^{(n)}, F_k^{(n)}) + A(w_k^{(n)}).$$

Step 2 – domesticate interferences

Consider now the L^p norms: triangle inequality + TF invariance yield

$$\|A(f^{(n)})\|_{L^p(\Omega)} \le \sum_{j=1}^{k} \|A(f_j)\|_{L^p(\Omega)} + R_k^{(n)},$$

where the remainder satisfies $\lim_{k\to\infty} \limsup_{n\to\infty} R_k^{(n)} = 0$:

- $\|A(\pi(z_j^{(n)})f_j, \pi(z_{j'}^{(n)})f_{j'})\|_{L^p(\Omega)} \to 0$ since $A(f,g) \in C_0(\mathbb{R}^{2d})$
- $\|A(F_k^{(n)}, w_k^{(n)})\|_{L^p(\Omega)} \lesssim_\Omega \|V_g w_k^{(n)}\|_\infty \to 0$ (technical)
- $\|A(w_k^{(n)})\|_{L^p(\Omega)} \to 0$ similarly

Step 3 – existence of a maximizer

Recall that $L := \sup \left\{ \frac{\|Af\|_{L^p(\Omega)}}{\|f\|_{L^2}^2} : f \in L^2(\mathbb{R}^d) \setminus \{0\} \right\}$.

Passing to limits in the profile decomposition yields

$$L = \lim_{n\to\infty} \|A(f^{(n)})\|_{L^p(\Omega)} \le \sum_{j=1}^{\infty} \|A(f_j)\|_{L^p(\Omega)}.$$

On the other hand, by the very definition of L, $\|A(f_j)\|_{L^p(\Omega)} \le L\|f_j\|_{L^2}^2$.

Therefore, we infer

$$L \le \sum_{j=1}^{\infty} \|A(f_j)\|_{L^p(\Omega)} \le L \sum_{j=1}^{\infty} \|f_j\|_{L^2}^2 \le L.$$

Inequalities must be equalities, hence $\|Af_j\|_{L^p(\Omega)} = L\|f_j\|_{L^2}^2$ **for all** j.

Any non-null profile f_j is then a maximizer – there is at least one.

Step 4 – approaching an optimizer

If $p > 1$, the stronger claim $\pi(-z^{(n)})f^{(n)} \to \tilde{f}$ in L^2 for suitable $z^{(n)}$ and subsequences requires additional work.

Asymptotic decoupling estimates for the ambiguity function must be refined via **asymptotic interpolation**: setting $p^* = \min\{p, p'\}$,

$$\limsup_{n\to\infty} \left\| \sum_{j=1}^{k} A(\pi(z_j^{(n)})f_j) \right\|_{L^p(\Omega)} \le \left(\sum_{j=1}^{k} \|A(f_j)\|_{L^p(\Omega)}^{p^*} \right)^{1/p^*} .$$

Arguing as above, for $p > 1$ inequalities are equalities iff all but one profile (say f_1) are null. Then, $f^{(n)} = \pi(z_1^{(n)})f_1 + w_1^{(n)}$ with $\|f_1\|_{L^2} = 1$.

Since $\pi(z_1^{(n)})^* f^{(n)} \rightharpoonup f_1$ by construction and $\|f^{(n)}\|_{L^2} = \|f_1\|_{L^2} = 1$ we infer that

$$\pi(-z_1^{(n)})f^{(n)} \to cf_1 \text{ in } L^2.$$

Related results

[Nicola, Romero, T. – *Calc. Var.*, 2023]

- **The case $p = \infty$**

  $\sup \left\{ \frac{\|Af\|_{L^\infty(\Omega)}}{\|f\|_{L^2}^2} : f \in L^2(\mathbb{R}^d) \setminus \{0\} \right\} = 1$ and it is attained iff

  $$|\Omega \cap B_r| > 0 \text{ for every } r > 0, \quad \text{where } B_r = \{z \in \mathbb{R}^{2d} : |z| < r\}.$$

  In this case, every $f \in L^2(\mathbb{R}^d) \setminus \{0\}$ is a maximizer.

- **Failure for separate time/frequency auto-correlation**

  $$\sup_{f \in L^2(\mathbb{R}^d)\setminus\{0\}} \frac{\left(\int_\Omega |\langle f, T_x f \rangle|^p dx \right)^{1/p}}{\|f\|_{L^2}^2} = |\Omega|^{1/p}$$

  and the supremum is not attained. Similarly for $|\langle f, M_\xi f \rangle|$.

Modulation spaces

For $0 < p \le \infty$ we introduce the **modulation spaces**

$$M^p(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{M^p} < \infty \right\},$$

where $\|f\|_{M^p(\mathbb{R}^d)} := \|\langle f, \pi(\cdot)g\rangle\|_{L^p(\mathbb{R}^{2d})}$ for some (hence any) $g \in \mathcal{S}(\mathbb{R}^d)$.

A *scale* of (quasi-)Banach spaces: if $p \le q$ then $M^p \subseteq M^q$.

Variations on a theme

Modulation spaces [Nicola, Romero, T. – *Calc. Var.*, 2023]

Let $\Omega \subset \mathbb{R}^{2d}$ be a measurable subset of finite measure $|\Omega| > 0$.

- **Existence of an ambiguity concentration optimizer**

  If $1 \le p < \infty$ and $0 < q < 2$, there exists $\tilde{f} \in M^q(\mathbb{R}^d)$ such that

$$\frac{\|A\tilde{f}\|_{L^p(\Omega)}}{\|\tilde{f}\|_{M^q}^2} = \sup\left\{ \frac{\|Af\|_{L^p(\Omega)}}{\|f\|_{M^q}^2} : f \in M^q(\mathbb{R}^d) \setminus \{0\} \right\}.$$

- **Approaching the optimizer via (sub)sequences**

  If $1 < p < \infty$ and $0 < q < 2$, for every M^q-normalized maximizing sequence $f^{(n)}$ there exist $z^{(n)} \in \mathbb{R}^{2d}$ such that

$$\pi(-z^{(n)})f^{(n)} \to \tilde{f} \quad \text{in} \quad M^q.$$

Comments

How to get profile decompositions in M^q?

- Concentration compactness in Banach spaces ($1 \le q < 2$)
- Handcrafting profiles ($q = 1$)
- Start from CC in L^2 (since $M^q \hookrightarrow L^2$), then prove that profiles actually belong to M^q with $\sum_j \|f_j\|_{M^q} \le 1$ – **non-trivial!**

Existence of optimizers holds (!) if **discrete norms** are used in M^q:

$$|f|_{M^q} := \Big(\sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda) g \rangle|^q \Big)^{1/q},$$

where the Gabor transform is sampled along a full-rank lattice $\Lambda \subset \mathbb{R}^{2d}$ such that $\{\pi(\lambda)g : \lambda \in \Lambda\}$ is a **Gabor frame** for L^2, that is

$$\sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda) g \rangle|^2 \asymp \|f\|_{L^2}^2 .$$

What next?

- **Shape of optimizers** – what about special domains?
- **Numerical explorations** – understand maximizing sequences
- **Other TF representations** – Wigner distribution (TF covariant)?
- **QHA** – UPs/optimization for mixed-state localization operators?

Thanks for your attention!

Find out more:

Nicola, Fabio; Romero, José Luis; Trapasso, S. Ivan.

On the existence of optimizers for time-frequency concentration problems.

Calc. Var. **62**, 21 (2023)

DOI: 10.1007/s00526-022-02358-6 – **arXiv:2112.09675**

Free access link to the paper in my **personal homepage**.